Problem 2.33

The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as follows:

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

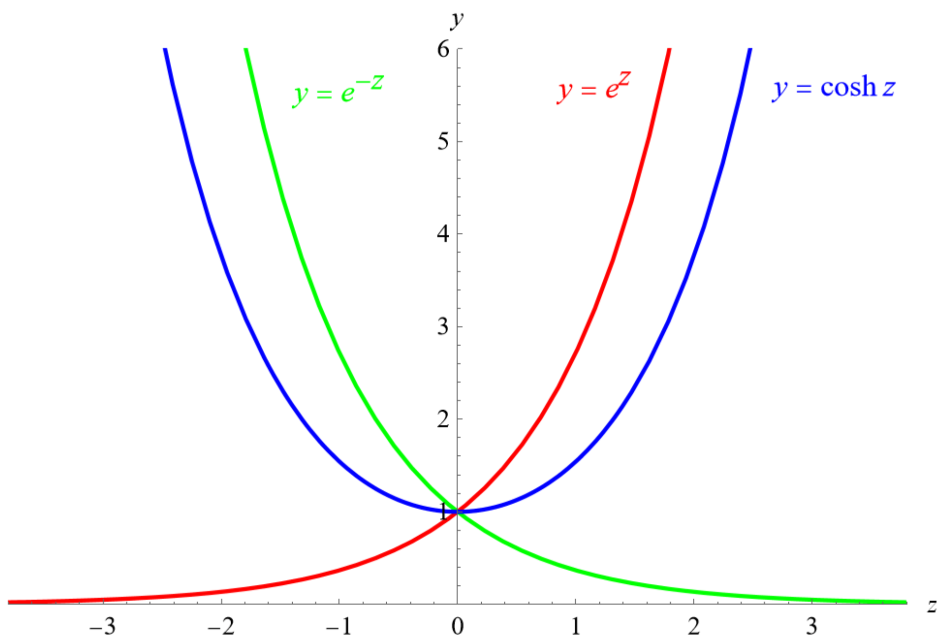
for any z, real or complex. **(a)** Sketch the behavior of both functions over a suitable range of real values of z. **(b)** Show that $\cosh z = \cos(iz)$. What is the corresponding relation for $\sinh z$? **(c)** What are the derivatives of $\cosh z$ and $\sinh z$? What about their integrals? **(d)** Show that $\cosh^2 z - \sinh^2 z = 1$. **(e)** Show that $\int dx/\sqrt{1+x^2} = \operatorname{arcsinh} x$. [*Hint:* One way to do this is to make the substitution $x = \sinh z$.]

[TYPO: Replace $\operatorname{arcsinh} x$ with $\operatorname{arcsinh} x + C$.]

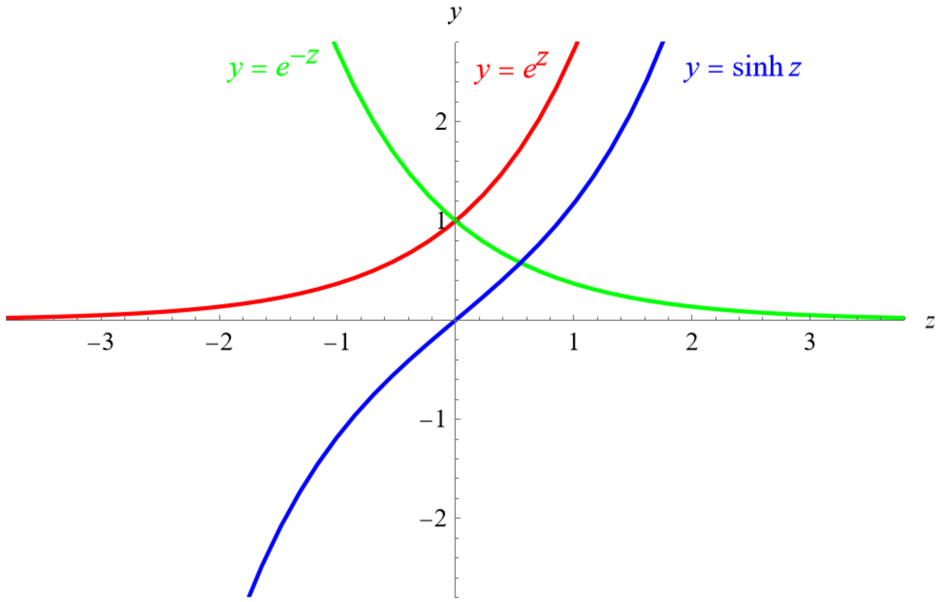
Solution

Part (a)

Each point on the graph of $\cosh z$ is obtained by taking the average of the corresponding points on the e^z and e^{-z} curves.



Each point on the graph of $\sinh z$ is half the distance from the e^{-z} curve to the e^z curve.



Part (b)

Begin with the definition of cosine in terms of exponential functions.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Replace z with iz.

$$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \frac{e^{-z} + e^{z}}{2}$$

$$= \frac{e^{z} + e^{-z}}{2}$$

$$= \cosh z$$

As a result,

$$\boxed{\cosh z = \cos iz.}$$

The definition of sine in terms of exponential functions is

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Replace z with iz.

$$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i}$$

$$= \frac{e^{-z} - e^{z}}{2i}$$

$$= -\frac{1}{i}\left(\frac{e^{z} - e^{-z}}{2}\right)$$

$$= -\frac{1}{i}\sinh z$$

Therefore, multiplying both sides by $-i$,

$$\boxed{\sinh z = -i\sin iz.}$$

Part (c)

Take the derivative of $\cosh z$.

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^{z} + e^{-z}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{d}{dz}(e^{z}) + \frac{d}{dz}(e^{-z})\right]$$

$$= \frac{1}{2}[(e^{z}) + (-e^{-z})]$$

$$= \frac{e^{z} - e^{-z}}{2}$$

$$= \sinh z$$

$$\boxed{\frac{d}{dz}\cosh z = \sinh z}$$

Take the derivative of $\sinh z$.

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^{z} - e^{-z}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{d}{dz}(e^{z}) - \frac{d}{dz}(e^{-z})\right]$$

$$= \frac{1}{2}[(e^{z}) - (-e^{-z})]$$

$$= \frac{e^{z} + e^{-z}}{2}$$

$$= \cosh z$$

Therefore,

$$\boxed{\frac{d}{dz}\sinh z = \cosh z.}$$

Take the integral of $\cosh z$.

$$\begin{aligned}
\int \cosh z\,dz &= \int \left(\frac{e^z + e^{-z}}{2}\right) dz \\
&= \frac{1}{2}\left(\int e^z\,dz + \int e^{-z}\,dz\right) \\
&= \frac{1}{2}[(e^z) + (-e^{-z})] + C \\
&= \frac{e^z - e^{-z}}{2} + C \\
&= \sinh z + C
\end{aligned}$$

As a result,

$$\boxed{\int \cosh z\,dz = \sinh z + C.}$$

Take the integral of $\sinh z$.

$$\begin{aligned}
\int \sinh z\,dz &= \int \left(\frac{e^z - e^{-z}}{2}\right) dz \\
&= \frac{1}{2}\left(\int e^z\,dz - \int e^{-z}\,dz\right) \\
&= \frac{1}{2}[(e^z) - (-e^{-z})] + C \\
&= \frac{e^z + e^{-z}}{2} + C \\
&= \cosh z + C
\end{aligned}$$

Therefore,

$$\boxed{\int \sinh z\,dz = \cosh z + C.}$$

Part (d)

Simplify $\cosh^2 z - \sinh^2 z$.

$$\begin{aligned}
\cosh^2 z - \sinh^2 z &= (\cosh z + \sinh z)(\cosh z - \sinh z) \\
&= \left(\frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2}\right)\left(\frac{e^z + e^{-z}}{2} - \frac{e^z - e^{-z}}{2}\right) \\
&= \left(\frac{e^z}{2} + \cancel{\frac{e^{-z}}{2}} + \frac{e^z}{2} - \cancel{\frac{e^{-z}}{2}}\right)\left(\cancel{\frac{e^z}{2}} + \frac{e^{-z}}{2} - \cancel{\frac{e^z}{2}} + \frac{e^{-z}}{2}\right) \\
&= (e^z)(e^{-z}) \\
&= e^{z-z} \\
&= 1
\end{aligned}$$

Therefore,

$$\boxed{\cosh^2 z - \sinh^2 z = 1.}$$

Part (e)

Evaluate the given integral.

$$\int \frac{dx}{\sqrt{1+x^2}} = \int^x \frac{dx'}{\sqrt{1+x'^2}} + C$$

Make the given substitution.

$$x' = \sinh z$$

$$dx' = \cosh z \, dz$$

As a result,

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+x^2}} &= \int^{\sinh^{-1} x} \frac{\cosh z \, dz}{\sqrt{1+\sinh^2 z}} + C \\
&= \int^{\sinh^{-1} x} \frac{\cosh z \, dz}{\sqrt{\cosh^2 z}} + C \\
&= \int^{\sinh^{-1} x} \frac{\cosh z \, dz}{\cosh z} + C \\
&= \int^{\sinh^{-1} x} dz + C \\
&= \sinh^{-1} x + C.
\end{aligned}$$

Therefore,

$$\boxed{\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh} x + C.}$$